MATH 310 — Quiz 3
25 March 2019

Full name: Solutions

This quiz is worth a total of **20 points**.

This quiz has 2 questions.

1. (10 points) For the following matrices, write the kernel and the image as the span of a finite number of vectors.

$$\text{(a)}\quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \text{(b)}\quad \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}.$$

(a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \xmapsto{R2-2R1} \begin{bmatrix} \boxed{1} & 2 \\ 0 & \boxed{-3} \end{bmatrix}$

For the image we have to take the columns of A for which we have the non-zero pivots

For the kernel we have $Ax=0$ $\quad \begin{matrix} -3x_2 = 0 \\ x_1 + 2x_2 = 0 \end{matrix} \quad \begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}$

Thus $\operatorname{Im}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ $\quad \ker(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

(b) $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \xmapsto[R3-3R1]{R2-2R1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \xmapsto{R3-2R2} \begin{bmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{-1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$

For the image we need to take the columns of A for which we have the non-zero pivots.

For the kernel we have $Ax=0$

$$x_1 + x_2 + 2x_3 = 0 \qquad x_2 = -x_3$$

$$-x_2 - x_3 = 0 \qquad x_1 - x_3 + 2x_3 = 0 \qquad x_1 = -x_3$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Thus, $\operatorname{Im}(A) = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

$\ker(A) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$.

Definition 1 *An inner product on the real vector space V is a pairing that takes two vectors $v, w \in V$ and produces a real number $< v, w > \in \mathbb{R}$. The inner product is required to satisfy the following three axioms for all $u, v, w \in V$, and scalars $c, d \in \mathbb{R}$*

1. *Bilinearity: $< cu + dv, w >= c < u, w > +d < v, w >$, and $< u, cv + dw >= c < u, v > +d < u, w >$*
2. *Symmetry: $< v, w >=< w, v >$*
3. *Positivity: $< v, v >> 0$ whenever $v \neq 0$, while $< 0, 0 >= 0$*

2. (10 points) Which of the following formulas for $< v, w >$ define inner products on $\mathbb{R}^2$. Explain your answer. You may use the definition above.
(a) $< v, w >= v_1 w_1 + 4 v_2 w_2$, (b) $< v, w >= (v_1 + v_2)(w_1 + w_2)$.

$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$.

(a) Bilinearity: $<cu+dv, w> = (cu_1 + dv_1)w_1 + 4(cu_2+dv_2)w_2 =$
$= c(u_1 w_1 + 4u_2 w_2) + d(v_1 w_1 + 4 v_2 w_2) = c<u,w> + d<v,w>$.
The same steps prove $<u, cv+dw> = c<u,v> + d<u,w>$.
Symmetry: $<v,w> = v_1 w_1 + 4 v_2 w_2 = w_1 v_1 + 4 w_2 v_2 = <w,v>$.
Positivity: $<v,v> = v_1 \cdot v_1 + 4 v_2 v_2 = v_1^2 + 4v_2^2$
when $v \neq 0$ $v_1^2 + 4v_2^2 > 0$ and when $v_1 = v_2 = 0$
$v_1^2 + 4v_2^2 = 0$.
thus $<u,v> = u_1 v_1 + 4u_2 v_2$ is an inner product

(b) We show that the Positivity condition violates.

$<v,v> = (v_1 + v_2)(v_1 + v_2) = (v_1 + v_2)^2$.

Note that when $v_1 = -v_2$ for example
$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $<v,v> = 0$ while $v \neq 0$.

Thus $<u,v> = (u_1 + u_2)(v_1 + v_2)$ is not an inner product.