Math 310, Applied Linear Algebra **Name:** Solutions
Spring 2019 **Midterm 3**
Date: 03/04/2019 **Time: 50 mins**

- **DO NOT open the exam booklet until you are told to begin. You should write your name at the top and read the instructions.**
- Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.
- You may use any results from class or the text, but you must cite the result you are using. You must prove everything else.
- This exam contains 5 numbered problems. The last page is blank. Check to see if any pages are missing. Point values are in parentheses.
- **No books, notes, or electronic devices are allowed.**

Problem	Points	Score
1	15	
2	25	
3	20	
4	20	
5	20	
Total:	100	

Definition 1. *An inner product on the real vector space V is a pairing that takes two vectors $\mathbf{v}, \mathbf{w} \in V$ and produces a real number $\langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{R}$. The inner product is required to satisfy the following three axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $c, d \in \mathbb{R}$:*

1. ***Bilinearity**: $\langle c\mathbf{u} + d\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{w} \rangle + d\langle \mathbf{v}, \mathbf{w} \rangle$, and $\langle \mathbf{u}, c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle + d\langle \mathbf{u}, \mathbf{w} \rangle$.*
2. ***Symmetry**: $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.*
3. ***Positivity**: $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ whenever $\mathbf{v} \neq 0$, while $\langle \mathbf{0}, \mathbf{0} \rangle = 0$.*

Definition 2. *A norm on a vector space V assigns a non-negative real number $||\mathbf{v}||$ to each vector $\mathbf{v} \in V$, subject to the following axioms, valid for every $\mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$:*

1. ***Positivity**: $||\mathbf{v}|| \geq 0$ with $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.*
2. ***Homogeneity**: $||c\mathbf{v}|| = |c| ||\mathbf{v}||$.*
3. ***Triangle inequality**: $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.*

1. (15 points) Which two of the vectors $\mathbf{v} = (1, 4, 1)^T$, $\mathbf{w} = (0, 0, -1)^T$ and $\mathbf{u} = (-2, 2, 1)^T$ are closest to each other in distance for (a) the Euclidean norm? (b) the ∞ norm (c) the 1 norm. Show your calculations.

2. (25 points) Characterize the image and kernel of the following matrices

$$\text{(a)} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \end{pmatrix}, \text{(b)} \begin{pmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 1 & 5 & 4 \end{pmatrix}.$$

3. (20 points) Prove that if $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle$ are two inner products on the same vector space V, then their sum $\langle\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle\rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle$ is an inner product on V. The definition of the inner product is on the front page.

4. (20 points) Let $\mathbf{K} = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$. Prove that the associated quadratic form $q(\mathbf{x}) = \mathbf{x}^T\mathbf{K}\mathbf{x}$ is indefinite by finding a point $\mathbf{x}_+$ where $q(\mathbf{x}_+) > 0$ and a point $\mathbf{x}_-$ where $q(\mathbf{x}_-) < 0$.

5. (20 points) Which of the following formulas define norm on $\mathbb{R}^3$. The definition of the norm is on the front page.

   For $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, (a) $||\mathbf{v}|| = \max(|v_1|, |v_2|, |v_3|)$, (b) $||\mathbf{v}|| = |v_1| + \max(|v_2|, |v_3|)$