Math 310, Applied Linear Algebra **Name:** Solutions
Spring 2019 **Midterm 3**
Date: 03/04/2019 **Time: 50 mins**

- **DO NOT open the exam booklet until you are told to begin. You should write your name at the top and read the instructions.**
- Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.
- You may use any results from class or the text, but you must cite the result you are using. You must prove everything else.
- This exam contains 5 numbered problems. The last page is blank. Check to see if any pages are missing. Point values are in parentheses.
- **No books, notes, or electronic devices are allowed.**

Problem	Points	Score
1	15	
2	25	
3	20	
4	20	
5	20	
Total:	100	

Definition 1. *An inner product on the real vector space V is a pairing that takes two vectors $\mathbf{v}, \mathbf{w} \in V$ and produces a real number $\langle \mathbf{v}, \mathbf{w} \rangle \in \mathbb{R}$. The inner product is required to satisfy the following three axioms for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $c, d \in \mathbb{R}$:*

1. ***Bilinearity**: $\langle c\mathbf{u} + d\mathbf{v}, \mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{w} \rangle + d\langle \mathbf{v}, \mathbf{w} \rangle$, and $\langle \mathbf{u}, c\mathbf{v} + d\mathbf{w} \rangle = c\langle \mathbf{u}, \mathbf{v} \rangle + d\langle \mathbf{u}, \mathbf{w} \rangle$.*
2. ***Symmetry**: $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$.*
3. ***Positivity**: $\langle \mathbf{v}, \mathbf{v} \rangle > 0$ whenever $\mathbf{v} \neq 0$, while $\langle \mathbf{0}, \mathbf{0} \rangle = 0$.*

Definition 2. *A norm on a vector space V assigns a non-negative real number $||\mathbf{v}||$ to each vector $\mathbf{v} \in V$, subject to the following axioms, valid for every $\mathbf{v}, \mathbf{w} \in V$ and $c \in \mathbb{R}$:*

1. ***Positivity**: $||\mathbf{v}|| \geq 0$ with $||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.*
2. ***Homogeneity**: $||c\mathbf{v}|| = |c|||\mathbf{v}||$.*
3. ***Triangle inequality**: $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.*

1. (15 points) Which two of the vectors $\mathbf{v} = (1,4,1)^T$, $\mathbf{w} = (0,0,-1)^T$ and $\mathbf{u} = (-2,2,1)^T$ are closest to each other in distance for (a) the Euclidean norm? (b) the ∞ norm (c) the 1 norm. Show your calculations.

$$v - w = \begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix} \quad v - u = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \quad w - u = \begin{pmatrix} 2 \\ -2 \\ -2 \end{pmatrix}$$

(a) $\|v-w\|_2 = \sqrt{1^2+4^2+2^2} = \sqrt{21}$

$\|v-u\|_2 = \sqrt{3^2+2^2+0^2} = \sqrt{13}$

$\|w-u\|_2 = \sqrt{2^2+2^2+2^2} = \sqrt{12}$

w and u are the closest ones by the Euclidean norm

(b) $\|v-w\|_\infty = \max\{1,4,2\} = 4$

$\|v-u\|_\infty = \max\{3,2,0\} = 3$

$\|w-u\|_\infty = \max\{2,2,2\} = 2$

w and u are the closest ones by the ∞ norm

(c) $\|v-w\|_1 = |1|+|4|+|2| = 7$

$\|v-u\|_1 = |3|+|2|+|0| = 5$

$\|w-u\|_1 = |2|+|2|+|2| = 6$

v and w are the closest ones by the 1-norm

2. (25 points) Characterize the image and kernel of the following matrices

$$\text{(a)} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \end{pmatrix}, \text{(b)} \begin{pmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 1 & 5 & 4 \end{pmatrix}.$$

(a)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 6 & 7 \end{bmatrix} \xrightarrow{R2-3R1} \begin{bmatrix} \boxed{1} & 2 & 3 \\ 0 & 0 & \boxed{-2} \end{bmatrix}$$

For the Kernel $-2z = 0, z=0$ and $x+2y+3z=0$,

$x+2y=0 \quad x=-2y, \quad \begin{pmatrix} -2y \\ y \\ 0 \end{pmatrix} = y\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \quad \ker(A) = \text{span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$.

$\text{Im}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 7 \end{pmatrix}\right\}$.

(b)

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 4 & 1 \\ 1 & 5 & 4 \end{bmatrix} \xrightarrow[R3-R1]{R2-3R1} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & -5 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{R3+\frac{2}{5}R2} \begin{bmatrix} \boxed{1} & 3 & 2 \\ 0 & \boxed{-5} & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Im}(A) = \text{span}\left\{\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}\right\}$.

For the Kernel

$-5y - 5z = 0 \qquad y = -z$

$x + 3y + 2z = 0 \qquad x - 3z + 2z = 0 \qquad x = z. \quad \begin{pmatrix} z \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

$\ker(A) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$.

3. (20 points) Prove that if $\langle \mathbf{u}, \mathbf{v} \rangle$ and $\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle$ are two inner products on the same vector space V, then their sum $\langle\langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle\rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle\langle \mathbf{u}, \mathbf{v} \rangle\rangle$ is an inner product on V. The definition of the inner product is on the front page.

We have to verify the 3 axioms of the inner product:

(1) Bilinearity: $\langle\langle\langle cu + dv, w \rangle\rangle\rangle = \langle cu + dv, w \rangle + \langle\langle cu + dv, w \rangle\rangle =$

$= c \langle u, w \rangle + d \langle v, w \rangle + c \langle\langle u, w \rangle\rangle + d \langle\langle v, w \rangle\rangle =$

$= c\left(\langle u, w \rangle + \langle\langle u, w \rangle\rangle \right) + d \left(\langle v, w \rangle + \langle\langle v, w \rangle\rangle \right) = c \langle\langle\langle u, w \rangle\rangle\rangle + d \langle\langle\langle v, w \rangle\rangle\rangle.$

Similarly $\langle\langle\langle u, cv + dw \rangle\rangle\rangle = c \langle\langle\langle u, v \rangle\rangle\rangle + d \langle\langle\langle u, w \rangle\rangle\rangle.$

(2) Symmetry:

$\langle\langle\langle u, v \rangle\rangle\rangle = \langle u, v \rangle + \langle\langle u, v \rangle\rangle = \langle v, u \rangle + \langle\langle v, u \rangle\rangle =$

$= \langle\langle\langle v, u \rangle\rangle\rangle$.

(3) Positivity: $\langle\langle\langle v, v \rangle\rangle\rangle = \langle v, v \rangle + \langle\langle v, v \rangle\rangle$

If $v \neq 0$ $\langle v, v \rangle > 0$ and $\langle\langle v, v \rangle\rangle > 0$

Thus $\langle\langle\langle v, v \rangle\rangle\rangle > 0$

If $v = 0$ $\langle\langle\langle v, v \rangle\rangle\rangle = 0 + = 0$.

4. (20 points) Let $\mathbf{K} = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix}$. Prove that the associated quadratic form $q(\mathbf{x}) = \mathbf{x}^T\mathbf{K}\mathbf{x}$ is indefinite by finding a point $\mathbf{x}_+$ where $q(\mathbf{x}_+) > 0$ and a point $\mathbf{x}_-$ where $q(\mathbf{x}_-) < 0$.

$q(x) = x^T K x$, let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$q(x) = (x_1 \quad x_2) \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1^2 + 6x_1x_2 + 4x_2^2$

Let $x_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ we get $q(x_+) = 11 > 0$.

Next, let $x_- = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ we get $q(x_-) = -1 < 0$.

5. (20 points) Which of the following formulas define norm on $\mathbb{R}^3$. The definition of the norm is on the front page.

For $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, (a) $||\mathbf{v}|| = \max(|v_1|, |v_2|, |v_3|)$, (b) $||\mathbf{v}|| = |v_1| + \max(|v_2|, |v_3|)$

We have to verify the 3 axioms of the norm: Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

(a) $||v|| = \max\{|v_1|, |v_2|, |v_3|\}$.

(1) Positivity: Since $|v_1| \geq 0$, $|v_2| \geq 0$ $|v_3| \geq 0$ then $\max\{|v_1|, |v_2|, |v_3|\} \geq 0$. If $||v|| = 0$, then $\max\{|v_1|, |v_2|, |v_3|\} = 0$ which implies that $|v_1| = 0$, $|v_2| = 0$ $|v_3| = 0$ thus $v_1 = v_2 = v_3 = 0$ and $v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

(2) Homogeneity: $cv = \begin{pmatrix} cv_1 \\ cv_2 \\ cv_3 \end{pmatrix}$

$$||cv|| = \max(|cv_1|, |cv_2|, |cv_3|) =$$

$$= \max(|c|\,|v_1|, |c|\,|v_2|, |c|\,|v_3|) = |c| \max(|v_1|, |v_2|, |v_3|) = ||v||.$$

(3) Triangle Inequality: $||u+v|| \leq ||u|| + ||v||$.

$u + v = \begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix}$. $||u+v|| = \max(|u_1+v_1|, |u_2+v_2|, |u_3+v_3|) \leq$

$$\leq \max(|u_1|+|v_1|, |u_2|+|v_2|, |u_3|+|v_3|) \leq \max(|u_1|, |u_2|, |u_3|) +$$

$$+ \max(|v_1|, |v_2|, |v_3|) = ||u|| + ||v||.$$

Thus $||v|| = \max(|v_1|, |v_2|, |v_3|)$ is a norm.

(b) $\|v\| = |v_1| + \max(|v_2|, |v_3|)$. $\quad v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$

(1) Positivity:

Since $|v_1|, |v_2|$ and $|v_3| \geq 0$ $\quad \|v\| = |v_1| + \max(|v_2|, |v_3|) \geq 0$

and if $\|v\| = 0$ $\quad |v_1| = |v_2| = |v_3| = 0$, thus $v_1 = v_2 = v_3 = 0$.

(2) Homogeneity. $\quad cv = \begin{pmatrix} cv_1 \\ cv_2 \\ cv_3 \end{pmatrix}$

$\|cv\| = |cv_1| + \max(|cv_2|, |cv_3|) = |c| \cdot (|v_1| +$

$+ \max(|v_2|, |v_3|)) = |c| \cdot \|v\|$.

(3) Triangle Inequality:

$\|u+v\| = |u_1+v_1| + \max(|u_2+v_2|, |u_3+v_3|) \leq$

$\leq |u_1| + |v_1| + \max(|u_2| + |v_2|, |u_3| + |v_3|) \leq$

$\leq |u_1| + \max(|u_2|, |u_3|) + |v_1| + \max(|v_2|, |v_3|) =$

$= \|u\| + \|v\|$

Thus $\|v\| = |v_1| + \max(|v_2|, |v_3|)$ is a norm.