Math 310, Applied Linear Algebra
Spring 2019
Date: 30/1/19

Name: Solutions
Midterm 1
Time: 50 mins

- **DO NOT open the exam booklet until you are told to begin. You should write your name at the top and read the instructions.**
- Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.
- You may use any results from class or the text, but you must cite the result you are using. You must prove everything else.
- This exam contains 5 numbered problems. The last sheet is blank. Check to see if any pages are missing. Point values are in parentheses.
- **No books, notes, or electronic devices are allowed.**

Problem	Points	Score
1	15	
2	15	
3	20	
4	20	
5	30	
Total:	100	

1. (15 points) Use Gaussian Elimination to solve the following system of linear equations

$$\begin{aligned} x_1 &+ 2x_2 &+ x_3 &= 1 \\ 3x_1 &+ x_2 &+ 2x_3 &= 4 \\ x_1 &+ 5x_2 &+ x_3 &= 10. \end{aligned}$$

We start with the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 3 & 1 & 2 & 4 \\ 1 & 5 & 1 & 10 \end{array}\right) \xrightarrow[R3-R1]{R2-3R1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 1 \\ 0 & 3 & 0 & 9 \end{array}\right) \longmapsto$$

$$\xrightarrow{R3+\frac{3}{5}R2} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 1 \\ 0 & 0 & -\frac{3}{5} & \frac{48}{5} \end{array}\right)$$

Now, by back substitution we get

$$-\frac{3}{5} X_3 = \frac{48}{5} \qquad X_3 = -16$$

$$-5X_2 - X_3 = 1 \qquad X_2 = 3$$

$$X_1 + 2X_2 + X_3 = 1 \qquad X_1 = 1 + 16 - 2\cdot 3 = -16$$

$$\boxed{X_1 = 11 \quad X_2 = 3 \quad X_3 = -16}$$

2. (a) (5 points) Under what conditions is the square A^2 of a matrix defined ?

The square of a matrix A is defined when A is a square matrix.

(b) (5 points) Show that A and A^2 commute.

We need to show that $A \cdot A^2 = A^2 \cdot A$.

Since matrix product is associative, we get that

$A \cdot A^2 = A(A \cdot A) = (A \cdot A)A = A^2 \cdot A.$

(c) (5 points) How many matrix multiplications are needed to compute A^n. Explain your answer.

To find A^n we can first calculate

$A^2 = A \cdot A$

$A^4 = A^2 \cdot A^2$

$\vdots$

$A^{2^k} = A^{2^{k-1}} \cdot A^{2^{k-1}}$

Now we can write

$n = \sum_{i=0}^{k} d_i 2^i$, where $d_i = 0$ or 1.

and $k = \lfloor \log_2 n \rfloor$.

$A^n = A^{\sum_{i=0}^{k} d_i 2^i}$.

3. Prove the following Lemmas:

(a) (5 points)

Lemma 1. *The inverse of a square matrix, if exists, is unique.*

Assume B and C are the inverses of A.
By the definition of the inverse $AB = BA = I$ and $AC = CA = I$.

by associativity of matrix product

Now, $B = B \cdot I = B(A \cdot C) = (B \cdot A) \cdot C = I \cdot C = C$.

Thus, $B = C$

(b) (5 points)

Lemma 2. *If A is an invertible matrix, then A^{-1} is also invertible and $(A^{-1})^{-1} = A$.*

We use the definition of the inverse
$A \cdot A^{-1} = A^{-1} \cdot A = I$ which means that A is the inverse of A^{-1}. Thus, $(A^{-1})^{-1} = A$.

(c) (10 points)

Lemma 3. *If A and B are invertible matrices of the same size, then their product, AB, is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.*

To show that $(AB)^{-1} = B^{-1}A^{-1}$, it is enough to show that $(AB)(B^{-1}A^{-1}) = I$ and $(B^{-1}A^{-1})(AB) = I$.

$(AB) \cdot (B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A \cdot I \cdot A^{-1} = AA^{-1} = I$

Also $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$.

4. (20 points) Find the inverse of the following matrix, by applying the Gauss-Jordan method.

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

We start with the augmented matrix

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xmapsto{R2-2R1} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & -2 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Next, we swap R2 & R3 to obtain

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \xmapsto{R4-\frac{2}{3}R2} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & -2 & 1 & 0 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & -\frac{2}{3} & 1 \end{array}\right]$$

Next, we swap R3 & R4 to obtain

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{4}{3} & 0 & 0 & 0 & -\frac{2}{3} & 1 \\ 0 & 0 & 0 & -2 & -2 & 1 & 0 & 0 \end{array}\right] \xmapsto[-\frac{R4}{2}]{\frac{R2}{3},\ \frac{3}{4}R3} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 \end{array}\right]$$

$$\xmapsto{R1-R4} \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 \end{array}\right] \xmapsto[R2-\frac{1}{3}R3]{R1-R3} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{4} \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & 1 & -\frac{1}{2} & 0 & 0 \end{array}\right]$$

Thus, the inverse is

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{4} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{2} & \frac{3}{4} \\ 1 & -\frac{1}{2} & 0 & 0 \end{bmatrix}.$$

5. (a) (20 points) Produce the LDV or a permuted LDV factorization of the following matrix:

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad A \mapsto \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix} \xrightarrow{R3-R1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & -3 & -2 \end{bmatrix} \xrightarrow{R3+\frac{3}{2}R2} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{5}{2} \end{bmatrix}.$$

$$U = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -\frac{3}{2} & 1 \end{bmatrix}$$

$$PA = LU$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \quad V = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix} \quad \boxed{PA = LDV}$$

(b) (10 points) Using the LDV factorization from part (a) solve the corresponding linear system $Ax = b$, for

$$A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 0 & 2 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We have $Ax = b$ $\quad PAx = Pb$ and $LDVx = Pb$.

$Lz = Pb$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$z_1 = 0 \quad z_2 = 1 \quad 0 - \frac{3}{2} + z_3 = 1$

$z_3 = \frac{5}{2}$

$Dy = z$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{5}{2} \end{bmatrix}$$

$y_1 = 0 \quad y_2 = \frac{1}{2} \quad y_3 = 1$

$Vx = y$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$$

$x_3 = 1 \quad x_2 + \frac{3}{2}x_3 = \frac{1}{2}, \; x_2 = -1$

$x_1 + 3x_2 + 4x_3 = 0 \quad x_1 = -1$

$$x = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$