MATH 310 — Quiz 4
24 April 2019

Full name: Solutions
This quiz is worth a total of **20 points**.

This quiz has 2 questions.

1. (10 points) Find the eigenvalues and eigenvectors of the following matrix.

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}.$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 & 1 \\ 1 & -\lambda & 1 \\ 2 & -2 & 3-\lambda \end{bmatrix}. \quad \det(A - \lambda I) = (2-\lambda)(-\lambda)(3-\lambda) - 2 -$$

$$-2 + 2\lambda - 2(2-\lambda) + (3-\lambda) = (2-\lambda)(-\lambda)(3-\lambda) + (3-\lambda) =$$

$$= (3-\lambda)(\lambda^2 - 2\lambda + 1) = (3-\lambda)(1-\lambda)^2$$

$\lambda_1 = 3$, $\lambda_2 = 1$ ← double eigenvalue.

For $\lambda_1 = 3$ $\quad A - 3I = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow[R3+2R1]{R2+R1} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -4 & 2 \\ 0 & -4 & 2 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} -1 & -1 & 1 \\ 0 & -4 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$v_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $\quad -4y + 2z = 0$, $y = \frac{z}{2}$, $-x - y + z = 0$, $-x - \frac{z}{2} + z = 0$

$$x = \frac{z}{2} \qquad \begin{pmatrix} \frac{z}{2} \\ \frac{z}{2} \\ z \end{pmatrix} = z \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}. \qquad v_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

For $\lambda = 1$ $\quad A - I = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & -2 & 2 \end{bmatrix} \xrightarrow[R3-2R1]{R2-R1} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$v_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$x - y + z = 0, \; x = y - z, \; \begin{pmatrix} y - z \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

2. (10 points) For matrix **A** from problem 1, which of the eigenvalues are complete? Is matrix **A** complete ? Is **A** diagonalizable? If so, find matrices **A** and Λ such that $\mathbf{A} = \mathbf{S}\Lambda\mathbf{S}^{-1}$. Explain your answers.

$\lambda_1 = 3$ the corresponding eigenvalue is $V_1 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}$. So λ_1 is complete.

$\lambda_2 = 1$ the corresponding eigenvalues are $V_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $V_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, thus $\lambda_2 = 1$ is also complete.

Since all eigenvalues of A are complete so is A. This implies that A is diagonalizable and

$$A = \begin{bmatrix} \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}.$$