Math 310, Applied Linear Algebra **Name:** Solutions

Spring 2019 **Midterm 2**

Date: 27/2/19 **Time: 50 mins**

- **DO NOT open the exam booklet until you are told to begin. You should write your name at the top and read the instructions.**
- Organize your work, in a reasonably neat and coherent way, in the space provided. If you wish for something to not be graded, please strike it out neatly. I will grade only work on the exam paper, unless you clearly indicate your desire for me to grade work on additional pages.
- You may use any results from class or the text, but you must cite the result you are using. You must prove everything else.
- This exam contains 5 numbered problems. The last page is blank. Check to see if any pages are missing. Point values are in parentheses.
- **No books, notes, or electronic devices are allowed.**

Problem	Points	Score
1	30	
2	15	
3	20	
4	15	
5	20	
Total:	100	

Definition 1. *A vector space is a set V equipped with two operations:*

1. *Addition: adding any pair of vectors $\mathbf{v}, \mathbf{w} \in V$ produces another vector $\mathbf{v} + \mathbf{w} \in V$;*
2. *Scalar Multiplication: multiplying a vector $\mathbf{v} \in V$ by a scalar $c \in \mathbb{R}$ produces a vector $c\mathbf{v} \in V$. These are subject to the following axioms, valid for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and all scalars $c, d \in R$:*

- *Commutativity of Addition:* $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.
- *Associativity of Addition:* $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.
- *Additive Identity: There is a zero element $\mathbf{0} \in V$ satisfying $\mathbf{v} + \mathbf{0} = \mathbf{v} = \mathbf{0} + \mathbf{v}$.*
- *Additive Inverse: For each $\mathbf{v} \in V$ there is an element $-\mathbf{v} \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0} = (-\mathbf{v}) + \mathbf{v}$.*
- *Distributivity:* $(c + d)\mathbf{v} = (c\mathbf{v}) + (d\mathbf{v})$, *and* $c(\mathbf{v} + \mathbf{w}) = (c\mathbf{v}) + (c\mathbf{w})$.
- *Associativity of Scalar Multiplication:* $c(d\mathbf{v}) = (cd)\mathbf{v}$.
- *Unit for Scalar Multiplication: the scalar $1 \in \mathbb{R}$ satisfies $1\mathbf{v} = \mathbf{v}$.*

1. Consider the following system of linear equations

$$\begin{aligned} x + 2y + 2z - 2u &= a, \\ 3x + 7y + z + 4u &= b, \\ -x - 3y + 4z + 3u &= c \end{aligned}$$

(a) (20 points) For which values of a, b and c the system above has no solution, one solution, infinitely many solutions

We start with the augmented matrix:

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -2 & a \\ 3 & 7 & 1 & 4 & b \\ -1 & -3 & 4 & 3 & c \end{array}\right) \xrightarrow[R3+R1]{R2-3R1} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -2 & a \\ 0 & 1 & -5 & 10 & b-3a \\ 0 & -1 & 6 & 1 & c+a \end{array}\right) \xrightarrow{R3+R2} \left(\begin{array}{cccc|c} 1 & 2 & 2 & -2 & a \\ 0 & 1 & -5 & 10 & b-3a \\ 0 & 0 & 1 & 11 & c+b-2a \end{array}\right)$$

As we can see, all three pivots are nonzero. Since there are

$$\begin{pmatrix} \boxed{1} & 2 & 2 & -2 \\ 0 & \boxed{1} & -5 & 10 \\ 0 & 0 & \boxed{1} & 11 \end{pmatrix}.$$

3 equations and the rank of the matrix is 3, for any choice of a, b and c the system always has infinitely many solutions.

(b) (10 points) Find the solution of the above system for

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

Let's plug in the values of a, b and c in the augmented matrix.

$$\left(\begin{array}{cccc|c} 1 & 2 & 2 & -2 & 3 \\ 0 & 1 & -5 & 10 & -8 \\ 0 & 0 & 1 & 11 & -4 \end{array}\right).$$

Thus, $z + 11u = -4$, if we let $u = t$ be the free variable, we get $z = -11t - 4$.

$y - 5z + 10u = -8$ $\quad y - 5(-11t - 4) + 10t = -8$, $y = -65t - 28$.

Finally, $x + 2y + 2z - 2u = 3$ $\quad x + 2(-65t - 28) + 2(-11t - 4) - 2t = 3$

$x = 154t + 67$.

The solution would be $\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 154t + 67 \\ -65t - 28 \\ -11t - 4 \\ t \end{pmatrix}$.

2. By using the following fact: $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$, prove the following propositions

(a) (5 points) Prove that $\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$

We use the fact that $I = A \cdot A^{-1}$. Thus,

$$\det(I) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}).$$

To conclude the proof, it is enough to note, that $\det(I) = 1$, which implies that $\det(A^{-1}) = \frac{1}{\det(A)}$.

(b) (5 points) Prove that $\det(\mathbf{AB}) = \det(\mathbf{BA})$.

$$\det(AB) = \det(A) \cdot \det(B) =$$

$$= \det(B) \cdot \det(A) = \det(BA).$$

(c) (5 points) Prove that $\det(\mathbf{AB}^{-1}) = \frac{\det(\mathbf{A})}{\det(\mathbf{B})}$.

$$\det(AB^{-1}) = \det(A) \cdot \det(B^{-1}) =$$

$$= \det(A) \cdot \frac{1}{\det(B)} = \frac{\det(A)}{\det(B)}.$$

3. Given the following vectors

$$\mathbf{v}_1 = \begin{bmatrix}1\\0\\2\\4\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}2\\0\\1\\5\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}0\\1\\3\\5\end{bmatrix}.$$

(a) (10 points) Check whether $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are independent or no.

To check independence it is enough to find the rank of the following matrix

$$\begin{bmatrix}1&2&0\\0&0&1\\2&1&3\\4&5&5\end{bmatrix} \xmapsto[R4-4R1]{R3-2R1} \begin{bmatrix}1&2&0\\0&0&1\\0&-3&3\\0&-3&5\end{bmatrix} \xmapsto{R2 \mapsto R3/3} \begin{bmatrix}1&2&0\\0&-1&1\\0&0&1\\0&-3&5\end{bmatrix} \xmapsto{R4-3R2} \begin{bmatrix}1&2&0\\0&-1&1\\0&0&1\\0&0&2\end{bmatrix} \mapsto$$

$$\xmapsto{R4-2R3} \begin{bmatrix}\boxed{1}&2&0\\0&\boxed{-1}&1\\0&0&\boxed{1}\\0&0&0\end{bmatrix}$$

As we can see the rank is 3. Thus, the vectors are linearly independent.

(b) (10 points) What is the span of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$? Is it equal to $\mathbb{R}^4$? Justify your answer.

Since V_1, V_2, V_3 are independent, their span would be a 3-dimensional subspace of $\mathbb{R}^4$.

No, 3 vectors can not span $\mathbb{R}^4$

4. (15 points) Determine whether the following set of vectors is a basis of R^3 or no.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}.$$

We need to check 2 things: linear independence and that $\text{span}\{v_1, v_2, v_3\} = \mathbb{R}^3$.

Since there are exactly 3 vectors, it is enough to calculate the rank of $[v_1\ v_2\ v_3]$ if it is 3, then v_1, v_2, v_3 would be a basis

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 5 & 0 & 5 \end{bmatrix} \xmapsto{R3-5R1} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -10 & 5 \end{bmatrix} \xmapsto{R3+10R2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 15 \end{bmatrix}$$

Thus, $\text{rank}([v_1\ v_2\ v_3]) = 3$.

Yes, v_1, v_2, v_3 form a basis of $\mathbb{R}^3$.

5. (20 points) A planar vector field is a function that assigns a vector

$$\mathbf{v}(x,y) = \begin{bmatrix} v_1(x,y) \\ v_2(x,y) \end{bmatrix}$$

to each point $(x,y) \in \mathbb{R}^2$. Prove that the set of all planar vector fields forms a vector space (the definition of the vector space is on the front page).

Let's denote the set of all planar vector fields by V.

We define summation and scalar multiplication on V as follows.

$u, v \in V$ $\underbrace{u(x,y) + v(x,y)}_{(u+v)(x,y)} = \begin{bmatrix} u_1(x,y) + v_1(x,y) \\ u_2(x,y) + v_2(x,y) \end{bmatrix}$ for all $(x,y) \in \mathbb{R}^2$

$c \in \mathbb{R}$

$(cv)(x,y) = c \cdot v(x,y) = \begin{bmatrix} c\, v_1(x,y) \\ c\, v_2(x,y) \end{bmatrix}$ for all $(x,y) \in \mathbb{R}^2$.

Now, we have to verify the 7 axioms.

(1) Commutativity of addition:

For all $u, v \in V$ $u + v = v + u$.

for all $(x,y) \in \mathbb{R}^2$ $u(x,y) + v(x,y) = \begin{bmatrix} u_1(x,y) + v_1(x,y) \\ u_2(x,y) + v_2(x,y) \end{bmatrix} = \begin{bmatrix} v_1(x,y) + u_1(x,y) \\ v_2(x,y) + u_2(x,y) \end{bmatrix} =$

$= v(x,y) + u(x,y)$. Thus, $u+v = v+u$.

(2) Associativity of addition

For all $u, v, w \in V$ $u + (v+w) = (u+v) + w$

For all $(x,y) \in \mathbb{R}^2$ $u(x,y) + (v(x,y) + w(x,y)) = \begin{bmatrix} u_1(x,y) + (v_1(x,y) + w_1(x,y)) \\ u_2(x,y) + (v_2(x,y) + w_2(x,y)) \end{bmatrix} =$

$= \begin{bmatrix} (u_1(x,y) + v_1(x,y)) + w_1(x,y) \\ (u_2(x,y) + v_2(x,y)) + w_2(x,y) \end{bmatrix} = (u(x,y) + v(x,y)) + w(x,y)$.

(3) Additive Identity:

for all $(x,y) \in \mathbb{R}^2$ $0(x,y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $u(x,y) + 0(x,y) = \begin{bmatrix} u_1(x,y)+0 \\ u_2(x,y)+0 \end{bmatrix} =$

$= \begin{bmatrix} u_1(x,y) \\ u_2(x,y) \end{bmatrix} = u(x,y)$. Similarly, for $0(x,y)+u(x,y)$.

(4) Additive Inverse:

For any $u \in V$ define $-u \in V$ as

$-u(x,y) = \begin{bmatrix} -u_1(x,y) \\ -u_2(x,y) \end{bmatrix}$ $u(x,y) + (-u)(x,y) = \begin{bmatrix} u_1(x,y) - u_1(x,y) \\ u_2(x,y) - u_2(x,y) \end{bmatrix} =$

$= \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Similarly, for $(-u)(x,y) + u(x,y)$.

(5) Distributivity: $(c+d)v = cv + dv$

$((c+d)v)(x,y) = \begin{bmatrix} (c+d)v_1(x,y) \\ (c+d)v_2(x,y) \end{bmatrix} = \begin{bmatrix} cv_1(x,y) \\ cv_2(x,y) \end{bmatrix} + \begin{bmatrix} dv_1(x,y) \\ dv_2(x,y) \end{bmatrix} = c\,v(x,y) + d\,v(x,y)$.

Similarly, for $c(u+w) = cu + cw$.

(6) Associativity for scalar multiplication: $c(d \cdot v) = (c \cdot d)v$

For all $(x,y) \in \mathbb{R}^2$, $(c(dv))(x,y) = \begin{bmatrix} c(d\,v_1(x,y)) \\ c(d\,v_2(x,y)) \end{bmatrix} = \begin{bmatrix} (cd)v_1(x,y) \\ (cd)v_2(x,y) \end{bmatrix} = ((cd)v)(x,y)$

(7) Unit for scalar multiplization $1 \cdot v = v$ for all $v \in V$.

$(1 \cdot v)(x,y) = \begin{bmatrix} 1 \cdot v_1(x,y) \\ 1 \cdot v_2(x,y) \end{bmatrix} = \begin{bmatrix} v_1(x,y) \\ v_2(x,y) \end{bmatrix} = v(x,y)$ for all $(x,y) \in \mathbb{R}^2$